

Non-commutative geometry and exactly solvable systems¹

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Abstract

I present the exact energy eigenstates and eigenvalues of a quantum many-body system of bosons on non-commutative space and in a harmonic oscillator confining potential at the selfdual point. I also argue that this exactly solvable system is a prototype model which provides a generalization of mean field theory taking into account non-trivial correlations which are peculiar to boson systems in two space dimensions and relevant in condensed matter physics. The prologue and epilogue contain a few remarks to relate my main story to recent developments in non-commutative quantum field theory and an addendum to our previous work together with Szabo and Zarembo on this latter subject.

Prologue

Recently the renormalizability of certain non-commutative (NC) quantum field theory (QFT) models was established in important work by Grosse and Wulkenhaar and the Orsay group [1–5]; see Ref. [6] for further references and a lucid discussion of the significance of these results. It seems that all examples of renormalizable such models share the duality property that their actions have the same form in Fourier- as in position space [7]. In this contribution I will discuss the following models possessing this latter property [8–11]:

$$\left. \begin{array}{l} \mathcal{H} \\ S \end{array} \right\} = \int_{\mathbb{R}^{2n}} d^{2n}x \left(\Phi^\dagger(x) [\sigma(-i\partial - B \cdot x)^2 + \tilde{\sigma}(-i\partial + B \cdot x)^2 - \mu] \Phi(x) + \tilde{g} \Phi^\dagger \star \Phi \star \Phi^\dagger \star \Phi(x) \right). \quad (1)$$

[Notation: I denote points in \mathbb{R}^{2n} , $2n = 2, 4, \dots$, by $x = (x^1, \dots, x^{2n})$; $\Phi^{(\dagger)}(x)$ represents a boson field to be specified in more detail below; I write

$$(-i\partial \pm B \cdot x)^2 \equiv \sum_{\mu=1}^{2n} \left(-i \frac{\partial}{\partial x^\mu} \pm (B \cdot x)_\mu \right)^2 \quad \text{with} \quad (B \cdot x)_\mu \equiv \sum_{\nu=1}^{2n} B_{\mu\nu} x^\nu \quad (2)$$

and $B = (B_{\mu\nu})_{\mu,\nu=1}^{2n}$ some fixed skewsymmetric and invertible $2n \times 2n$ matrix; $\sigma, \tilde{\sigma} \geq 0$, μ and \tilde{g} are real parameters such that $\sigma + \tilde{\sigma} > 0$; \star is the well-known Groenewold-Moyal product (see e.g. [12] for review) characterized by another skewsymmetric $2n \times 2n$ matrix $\theta = (\theta_{\mu\nu})_{\mu,\nu=1}^{2n}$ as follows,

$$x^\mu \star x^\nu - x^\nu \star x^\mu = -2i\theta^{\mu\nu}; \quad (3)$$

the dagger indicates complex conjugation and the Hilbert space adjoint; \mathbb{N} are the positive integers.] I denote the same mathematical expression by two different symbols since it can be either interpreted as the Hamiltonian (\mathcal{H}) of a quantum many-body system on $2n$ dimensional space [8, 9] or, alternatively, as an action (S) of a NC QFT model on $2n$ dimensional Euclidean spacetime [7, 10, 11]. My discussion will be mainly restricted to the special case where $B\theta = I$ (i.e. θ is the inverse of the matrix B) where these models are exactly solvable [8, 11]. One of my aims is to present results which have remained unpublished up to now, another to add a few remarks to our previous publications on this exactly solvable NC QFT model together with Szabo and Zarembo [10, 11] to point out an interesting alternative interpretation of our results, and to emphasize that an interesting problem concerning this model still remains to be solved. The new results are on the exactly solvable quantum many-body system of bosons and presented in sections 1–4, and my remarks on the NC QFT model are contained in the epilogue at the end.

Before going into my main story I shortly recall the simplification arising at $B\theta = I$ [8] (in this discussion I will refer to the mathematical object in (1) as Hamiltonian \mathcal{H} , but everything I say applies word-by-word also to its interpretation as action S). Obviously, the Hamiltonian in (1) is the sum of two term,

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$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{int}$, where \mathcal{H}_0 and \mathcal{H}_{int} are the quadratic and quartic parts in the fields $\Phi^{(\dagger)}$, respectively. As is well-known, in many standard field theory models one can expand the fields in a basis such that either the quadratic or the quartic part of the Hamiltonian becomes simple, but in general it is not possible to make both parts simple in the same basis, and this is one main reason why, in general, field theory models are computationally challenging (typically \mathcal{H}_0 is simple in Fourier space and \mathcal{H}_{int} in position space). However, for the model in (1) at the special point $B\theta = I$ there exists a basis in which both, \mathcal{H}_0 and \mathcal{H}_{int} , are simple. To be specific: this latter basis is given by the common eigenfunctions $\phi_{\ell m}(x)$ of the differential operators in (2) labeled by two sets of integer vectors $\ell, m \in \mathbb{N}^n$, and by expanding the fields in this basis

$$\Phi(x) = \sum_{\ell, m} A_{\ell m} \phi_{\ell m}^\dagger(x), \quad \Phi^\dagger(x) = \sum_{\ell, m} A_{\ell m}^\dagger \phi_{\ell m}(x) \quad (4)$$

the Hamiltonian acquires the following remarkably simple form [8, 11],

$$\left. \begin{array}{l} \mathcal{H} \\ S \end{array} \right\} = \sum_{\ell, m} (E_\ell + \tilde{E}_m - \mu) A_{\ell m}^\dagger A_{\ell m} + g \sum_{\ell, m, \ell', m'} A_{\ell m'}^\dagger A_{\ell m} A_{\ell' m}^\dagger A_{\ell' m'}. \quad (5)$$

The parameters E_ℓ and \tilde{E}_m are proportional to the eigenvalues of the operators in (2) and given by

$$E_\ell = \sum_{j=1}^n 4\sigma |B_j| (\ell_j - \frac{1}{2}), \quad \tilde{E}_m = \sum_{j=1}^n 4\tilde{\sigma} |B_j| (m_j - \frac{1}{2}) \quad (6)$$

with $\ell = (\ell_1, \ell_2, \dots, \ell_n)$ and similarly for m ; $|B_j|$ are eigenvalues of the matrix $\sqrt{B^2}$ (see [11] for a precise statement), and

$$g = \frac{\tilde{g}}{\sqrt{\det(4\pi\theta)}}. \quad (7)$$

It is interesting to note that the model in (5) can be written in the following matrix form,

$$\left. \begin{array}{l} \mathcal{H} \\ S \end{array} \right\} = \text{Trace}(EA^\dagger A + \tilde{E}AA^\dagger - \mu A^\dagger A + g(A^\dagger A)^2) \quad (8)$$

where A , E and \tilde{E} above stands for the infinite matrices with matrix elements $A_{\ell m}$, $E_\ell \delta_{\ell m}$, and $\tilde{E}_m \delta_{\ell m}$, respectively, the matrix adjungation is defined such that $(A^\dagger)_{\ell m} = A_{m\ell}^\dagger$, and matrix multiplication is understood.

1 Introduction

Interacting boson systems have been of interest in theoretical physics since the early days of quantum physics, and a recent increased interest in this subject was triggered by remarkable experimental progress to realize and study the Bose-Einstein condensation; see e.g. [13] for a recent text book in this topic. I believe that these developments provide a good additional motivation for studying the NC quantum many-body Hamiltonian \mathcal{H} in (5): as I will argue in more detail in my first remark in section 4, this Hamiltonian defines a prototype model which allows to study a particular type of correlations and its effect on the Bose-Einstein condensation in an exact solution.

In the main part of this paper I thus interpret the model in (1) as Hamiltonian \mathcal{H} of bosons moving on $2n$ dimensional space \mathbb{R}^{2n} and interacting with a particular four point interaction. I will show that this model is exactly solvable in the sense that all its energy eigenstates and eigenvalues can be computed explicitly. As will be seen, this exact solution provides an example of a correlated boson system. To simplify notation and to allow for a simple physical interpretation I restrict my discussion to the case $2n = 2$ and $\sigma = \tilde{\sigma} = 1/4$, but my results can be straightforwardly generalized to $2n > 2$ and general parameter values. The parameter μ corresponds to the chemical potential, and, for my purposes, one can assign to it any convenient value.

I mention in passing that the fermion variant of this model was introduced and analyzed in [8, 9] but, to my knowledge, the boson story presented here has not appeared in the literature before.

The plan of the rest of this paper is as follows. In section 2 I give a precise definition of the quantum many-body model, and in section 3 I present its solution. Section 4 contains various remarks, and, in particular, I explain there why I believe that this model is a prototype model for interacting bosons and relevant in condensed matter physics.

2 Definition of the model

I consider the quantum many-body system defined by the Hamiltonian in (1) where the boson fields $\Phi^{(\dagger)}(x)$ are operators acting on a boson Fock space \mathcal{F} defined by the usual canonical commutator relations and a normalized vacuum state Ω annihilated by all operators $\Phi(x)$; see e.g. [14]. Expanding the fields as in (4) these latter relations are equivalent to

$$[A_{\ell m}, A_{\ell' m'}^\dagger] = \delta_{\ell\ell'} \delta_{mm'}, \quad [A_{\ell m}, A_{\ell' m'}] = 0, \quad A_{\ell m} \Omega = 0 \quad (9)$$

for all ℓ, m, ℓ', m' , as usual. Choosing $B\theta = I$ this Hamiltonian can be written as

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{int}, \quad \mathcal{H}_0 = \sum_{\ell, m} (E_\ell + \tilde{E}_m) A_{\ell m}^\dagger A_{\ell m}, \quad \mathcal{H}_{int} = g \sum_{\ell, m, \ell', m'} A_{\ell m'}^\dagger A_{\ell' m}^\dagger A_{\ell m} A_{\ell' m'} \quad (10)$$

where I found it convenient to normal order the interaction term \mathcal{H}_{int} (this corresponds to a renormalization of μ which can be ignored) and rename $E_\ell - \mu$ to E_ℓ . To simplify my discussion I set $2n = 2$, $\sigma = \bar{\sigma} = 1/4$ and $\mu = |B|$ so that $\ell, m \in \mathbb{N}$ and

$$E_\ell + \tilde{E}_m = |B|(\ell + m - 2). \quad (11)$$

Then the model describes interacting bosons on two dimensional space and confined by harmonic oscillator potential, with $\ell - 1$ and $m - 1$ the usual harmonic oscillator quantum numbers. A useful alternative interpretation of the quantum numbers ℓ and m is as x - and y components of (quasi-)momenta of bosons in two dimensions. In this latter interpretation one is interested in other dispersion relations like $E_\ell + \tilde{E}_m \propto (\ell^2 + m^2)$ and ℓ, m running also over negative integers (see the first remark in section 4), and it is therefore important to note that such changes do not affect the exact solubility of the model. Then the interaction term describes two body scattering processes where two bosons with initial momenta (ℓ, m) and (ℓ', m') exchange the y -components of their momenta while the x -components remain the same, or vice versa [9].

3 Exact solution

I now discuss how to construct exact energy eigenstates of this model. For that I consider the quasi-free states (the normalization of the eigenstates will be ignored)

$$\eta = A_{\ell_1 m_1}^\dagger A_{\ell_2 m_2}^\dagger \cdots A_{\ell_N m_N}^\dagger \Omega \quad (12)$$

for fixed quantum numbers ℓ_j and m_j in \mathbb{N} , with N an arbitrary fixed non-negative integer. One can interpret this as a state containing N bosons with momenta (ℓ_j, m_j) . Each such state is an eigenstate of the quadratic part \mathcal{H}_0 of the Hamiltonian, and the corresponding eigenvalue is

$$\mathcal{E}_0 = \sum_{j=1}^N (E_{\ell_j} + \tilde{E}_{m_j}). \quad (13)$$

I will refer to this as kinetic energy.

It is important to note that the permutation group S_N of N elements has a natural action on these states η as follows,

$$P\eta := A_{\ell_1 m_{P1}}^\dagger A_{\ell_2 m_{P2}}^\dagger \cdots A_{\ell_N m_{PN}}^\dagger \Omega \quad (14)$$

for all $P \in S_N$, and that all these states $P\eta$ are degenerate eigenstates of \mathcal{H}_0 . Moreover, one can show that the action of the interaction part of the Hamiltonian \mathcal{H}_{int} on such a state η is

$$\mathcal{H}_{int}\eta = 2g \sum_{1 \leq j < k \leq N} T_{jk}\eta \quad (15)$$

where $T_{jk} \in S_N$ is the transposition which interchanges j and k and leaves all other integers $1, 2, \dots, N$ the same. One can interpret T_{jk} as the operator exchanging the y -components of the momenta of the j -th and the k -th boson leaving the x -components the same, or vice versa. Obviously this implies that all eigenstates of \mathcal{H} are of the form

$$\Psi = \sum_{P \in S_N} a_P P\eta \quad (16)$$

for some η and certain coefficients a_P to be determined. I will get back to the problem of how to construct all these eigenstates and corresponding eigenvalues further below.

For now I consider particular such eigenstates which can be obtained by elementary methods and which include the groundstates in the weak- and strong coupling limits. These eigenstates are given by

$$\eta^\pm = \sum_{P \in S_N} (\pm)^P P\eta \quad (17)$$

where $(+)^P$ is always 1 and $(-)^P = 1$ for even and -1 for odd permutations P , respectively. To see that these are eigenstates we note that $T_{jk}\Psi_\pm = \pm\Psi_\pm$, which implies Ψ^\pm is an exact eigenstate of \mathcal{H}_{int} with eigenvalue $\pm gN(N-1)$ (since $\sum_{1 \leq j < k \leq N} = N(N-1)/2$), and thus

$$\mathcal{H}\eta^\pm = (\mathcal{E}_0 \pm gN(N-1))\eta^\pm. \quad (18)$$

It is interesting to note that the state η^- has a fermion-like character and, as discussed below, this implies a strong variant of the Pauli exclusion principle which will play an important role. As will be shown further below, the states η^\pm are extremal in the sense that they have the largest possible interaction energies.

In particular, for $g \leq 0$ the groundstate of the model at fixed particle number N is the state η^+ such that the kinetic energy \mathcal{E}_0 assumes its smallest possible value. It is easy to see that the state in (12) with the minimum kinetic energy is

$$\eta_1 = (A_{1,1}^\dagger)^N \Omega, \quad (19)$$

i.e. all bosons are in the same one-particle state with momentum $(\ell, m) = (1, 1)$. Note that η_1 is the well-known Bose-Einstein condensate (BEC) groundstate of the non-interacting system ($g = 0$). In fact, this state is the groundstate for all $g < 0$ (this is true since η_1^+ equals η_1 up to a constant), and it is easy to see that the corresponding groundstate energy is

$$\mathcal{E}_1 = gN(N-1). \quad (20)$$

For $g > 0$ the states η^+ have a large interaction energy, and for sufficiently large $g > 0$ the groundstate of the model should be the state η^- with η such that the kinetic energy \mathcal{E}_0 is minimal. It is important to note that one now cannot take as η the state η_1 in (19) since η_1^- vanishes. More generally, the following strong variant of the Pauli exclusion principle holds true: *The state η^- in (17) is non-zero only if all the x - and all the y -components ℓ_j and m_j of the boson momenta in the state η in (12) are distinct.*² [Proof: Consider a state η in (12) such that $\ell_j = \ell_k$ and/or $m_j = m_k$ for some $j < k$. This implies $T_{jk}\eta = \eta$, but then $\eta^- = \sum_P (-)^P P\eta = \sum_P (-)^P PT_{jk}\eta = -\eta^-$, and thus $\eta^- = 0$.] A state whose momenta are all distinct and which has the lowest possible kinetic energy is

$$\eta_2 = A_{1,1}^\dagger A_{2,2}^\dagger \cdots A_{N,N}^\dagger \Omega, \quad (21)$$

and thus

$$\eta_2^- = \sum_{P \in S_N} (-1)^P A_{1,P1}^\dagger A_{2,P2}^\dagger \cdots A_{N,PN}^\dagger \Omega \quad (22)$$

is the groundstate of the model in the strong coupling limit. The corresponding groundstate energy is

$$\mathcal{E}_2 = |B|N(N-1) - gN(N-1) \quad (23)$$

(since $\sum_{\ell=1}^N (\ell-1) = N(N-1)/2$). As discussed below, η_2^- is actually the groundstate of the model not only in the strong coupling limit but for all $g \geq |B|$.

I now discuss the problem of finding the groundstate for intermediate coupling values. Note that, for fixed η in (12), the states $P\eta$ in (14), $P \in S_N$, span a subspace \mathcal{F}_η of the boson Fock space \mathcal{F} . The dimension of this subspace is $\leq N!$, and it is $N!$ if and only if all the x - and y -components ℓ_j and m_j of the bosons in the state η are distinct. It is important to note that (14) defines a representation of the permutation group S_N on \mathcal{F}_η , and this representation is, in general, reducible. Moreover, the operator

$$C_N = \sum_{1 \leq j < k \leq N} T_{jk}$$

²The standard Pauli principles for fermions is weaker since it only requires that all the momenta (ℓ_j, m_j) are distinct.

appearing in (15) commutes with all permutations $P \in S_N$, and it is therefore a constant in each irreducible representation (irrep) of S_N . Since on \mathcal{F}_η the kinetic energy \mathcal{H}_0 is constant and the interaction \mathcal{H}_{int} proportional to C_N , *the problem of constructing eigenstates of \mathcal{H} is equivalent to decomposing the representation of S_N on \mathcal{F}_η described above into irreps.* This is a classical problem solved in group theory; see e.g. Chapter IV in [15]: the irreps of S_N can be labeled by partitions λ of N , i.e. $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_K)$ with integers λ_j such that

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_K > 0, \quad \sum_{j=1}^K \lambda_j = N, \quad (24)$$

and the value of C_N in an irrep λ is $C_N = \sum_{j=1}^K [\frac{1}{2} \lambda_j (\lambda_j + 1) - j \lambda_j]$; see e.g. Eq. (4-3) in [16]. Moreover, the states in an irrep λ can be obtained by applying to states η the so-called Young symmetrizer [15] denoted by Y^λ . One thus concludes that the energy eigenstates of this model are $Y^\lambda \eta$ with the corresponding eigenvalues

$$\mathcal{E} = \sum_{j=1}^N (E_{\ell_j} + \tilde{E}_{m_j}) + \sum_{j=1}^K g \lambda_j (\lambda_j + 1 - 2j). \quad (25)$$

The states η^+ and η^- in (17) correspond to the special cases $\lambda = (N)$ and $\lambda = (1, 1, \dots, 1) \equiv (1^N)$, respectively. In principle this gives all eigenstates and eigenvalues of the model. There is, however, an important complication: as seen in the previous section for the special case $\lambda = (1^N)$, if there are degeneracies many of the eigenstates $Y^\lambda \eta$ vanish. To find the groundstate of the model we therefore must determine the state η in (12) of minimal kinetic energy and such that, for a fixed partition λ , $Y^\lambda \eta$ is non-zero. This problem has the following solution,

$$\eta = (A_{1,1}^\dagger)^{\lambda_1} (A_{2,2}^\dagger)^{\lambda_2} \dots (A_{K,K}^\dagger)^{\lambda_K} \Omega, \quad (26)$$

and the smallest possible energy eigenvalue in an irreps λ is therefore

$$\mathcal{E}_\lambda = \sum_{j=1}^K \lambda_j \left(E_j + \tilde{E}_j + g(\lambda_j + 1 - 2j) \right). \quad (27)$$

One can determine the groundstate of the model by finding the partition λ of N which minimizes the energy in (27). The solution of this problem depends on the dispersion relation $E_\ell + \tilde{E}_m$. Using the one in (11) one finds $\lambda = (N)$ for $g \leq 0$ and $\lambda = (1^N)$ for $g \geq |B|$, which confirms that the states in (19) and (21) are the groundstates for $g \leq 0$ and $g \geq |B|$, respectively. In the intermediate regime $0 < g < |B|$ the groundstate is given by a partition approximated by

$$\lambda_j \simeq \alpha(K + 1 - j), \quad K \simeq \sqrt{\frac{2N}{\alpha}} \text{ with } \alpha \simeq \frac{|B|}{g} - 1 > 0 \quad (28)$$

where " \simeq " means that the l.h.s. is the non-negative integer closest to the r.h.s., and this approximation becomes exact in the limit when the boson number N becomes infinite.

I finally note that one can prove that the eigenstates of this model are, in general, correlated by finding one non-zero connected 4-point correlation function. As an example I consider the normalized strong coupling groundstate for $N = 2$:

$$\Psi = \frac{1}{\sqrt{2}} \left(A_{1,1}^\dagger A_{2,2}^\dagger - A_{1,2}^\dagger A_{2,1}^\dagger \right) \Omega, \quad (29)$$

which supports the following non-trivial connected 4-point correlation function,

$$(\Psi, A_{1,2}^\dagger A_{2,1}^\dagger A_{1,1} A_{2,2} \Psi) - (\Psi, A_{1,2}^\dagger A_{1,1} \Psi)(\Psi A_{2,1}^\dagger A_{2,2} \Psi) - (\Psi, A_{1,2}^\dagger A_{2,2} \Psi)(\Psi A_{2,1}^\dagger A_{1,1} \Psi) = -\frac{1}{2} \quad (30)$$

with (\cdot, \cdot) the inner product in the boson Fock space.

4 Concluding remarks

1. A key problem in theoretical physics is to do reliable computations in quantum models with interactions so large that perturbation theory does not apply. One well-known and often successful strategy in this context is mean field theory. It is interesting to note that one approach to mean field theory is to truncate the interaction in the model under consideration and only keep the so-called Hartree- and Fock terms, which typically leads to an exactly soluble model whose solution is equivalent to mean field theory of the original model; see [9] for a discussion of this in the context of interacting fermion systems. Mean field theory does not take into account correlations, and the latter are believed to be particularly important in two spatial dimensions (2D). *I propose that the Hamiltonian in (10) defines a prototype model allowing to study important 2D correlations in an exact solution.* To motivate this I consider the following standard 2D boson model

$$\mathcal{H} = \sum_{\mathbf{k}} \frac{\mathbf{k}^2}{2M} b^\dagger(\mathbf{k}) b(\mathbf{k}) + \frac{U}{L^2} \sum_{\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4} \delta_{\mathbf{k}_1 + \mathbf{k}_2, \mathbf{k}_3 + \mathbf{k}_4} b^\dagger(\mathbf{k}_1) b^\dagger(\mathbf{k}_2) b(\mathbf{k}_3) b(\mathbf{k}_4) \quad (31)$$

with the boson mass $M > 0$ and coupling parameter $U > 0$. The boson operators $b^{(\dagger)}(\mathbf{k})$ are labeled by 2D momenta

$$\mathbf{k} = (k_x, k_y), \quad k_x, k_y \in \frac{2\pi}{L}\mathbb{Z} \quad \text{such that} \quad |k_{x,y}| < \frac{\pi}{a} \quad (32)$$

and obey the usual relations, $[b(\mathbf{k}), b^\dagger(\mathbf{k}')] = \delta_{\mathbf{k}, \mathbf{k}'}$ etc. The parameters $L \gg a > 0$ correspond to the system size (L) and a lattice constant (a) and provide a IR and UV cutoff for the model. The interaction term in this Hamiltonian comes from a local interaction in position space and describes scattering processes where two bosons with momenta \mathbf{k}_3 and \mathbf{k}_4 are scattered into states with momenta \mathbf{k}_1 and \mathbf{k}_2 , and the model is complicated since all possible such scattering processes occur with equal strength and are restricted only by overall momentum conservation. The Hartree- and Fock terms correspond to the scattering terms where $\mathbf{k}_1 = \mathbf{k}_4$, $\mathbf{k}_2 = \mathbf{k}_3$ and $\mathbf{k}_1 = \mathbf{k}_3$, $\mathbf{k}_2 = \mathbf{k}_4$, and they are (essentially) trivial for this model in the sense that they only add an energy $\simeq 4gN^2$ and do not (much) affect the groundstate. Note that the interaction contains also the scattering terms where

$$(k_1)_x = (k_4)_x, \quad (k_2)_x = (k_3)_x, \quad (k_1)_y = (k_3)_y, \quad (k_2)_y = (k_4)_y \quad (33)$$

and similar terms with x and y interchanged. These scattering terms are Hartree-like in the x - and Fock-like in the y -component of the momenta and vice versa, and they are peculiar to 2D. *If one restricts the interaction terms in the Hamiltonian in (31) and only includes these latter mixed Hartree-Fock terms one obtains exactly a Hamiltonian as in (10) with*

$$E_\ell + \tilde{E}_m = \frac{1}{2M} \left(\frac{2\pi}{L} \right)^2 (\ell^2 + m^2), \quad g = \frac{2U}{L^2} \quad (34)$$

and integers ℓ, m such that $|\ell|, |m| < L/(2a)$. As mentioned, this latter truncation is very similar to a successful method to derive useful mean field theories for interacting fermion systems, and I thus regard the model in (10) as a generalized mean field model. The exact solution of this model above does not rely on the form of $E_\ell + \tilde{E}_m$ (except for the groundstate, of course), and, as argued below, this model describes interesting “physics” which cannot be accounted for in mean field theory.

2. It is interesting to note how the character of the groundstate of the model changes with increasing coupling constant g : for $g = 0$ one has the standard BEC groundstate in (19) where all bosons are in the same one particle state $(\ell_j, m_j) = (1, 1)$. As the coupling increases it becomes more favorable to reduce the degeneracies and thus the number of bosons in the BEC, and one finds a distribution of the momenta as described by the partition in (28) and a correlated groundstate. Moreover, the BEC condensate in the ground state for $2|B|/(N+2) < g < |B|$ is

$$\langle A_{1,1}^\dagger A_{1,1} \rangle \simeq K\alpha \simeq \sqrt{2N \left(\frac{|B|}{g} - 1 \right)}, \quad (35)$$

and it becomes 1 for $g \geq |B|$ where the groundstate becomes maximally correlated. Moreover, as demonstrated in (30) above in a simple example, one can construct non-trivial 4-point Green's

functions for the model to prove that its groundstate is, in general, correlated and thus not accessible by mean field theory. I hope that these remarks are sufficient to convince the reader that the “physics” of this model is non-trivial and interesting. I plan to present a more detailed discussion elsewhere.

3. It is important to note that the model in (10) describes a stable system only in the parameter regime $0 \leq g \leq |B|$ since otherwise the groundstate energy can be decreased by increasing the particle number N to infinity (this follows from (20) and (23)). In my interpretation of this model as generalized Hartree-Fock model the instability for $g > |B|$ is removed by the Hartree-Fock energy $\simeq 4gN^2$ which should also be included.
4. Obviously nearly everything I wrote in the previous section can be immediately generalized to $2n > 2$ and other values for σ and $\tilde{\sigma}$, and the only change will be the solution to the problem to minimize the energy in (27).
5. In this paper I only computed the groundstate of the model and demonstrated how to compute the other energy eigenstates and eigenvalues. Obviously it would be interesting to compute also other quantities, like Green’s functions and the partition function.

Epilog

The two models in (1) are closely related: the NC QFT model defined by S can be obtained as infinite temperature limit of the quantum many-body model \mathcal{H} . Indeed, one can write the generating function for the Green’s functions of the latter model as matrix path integral (where the integration variables $A_{\ell m}^{(\dagger)}(\tau)$ are periodic functions of the Matsubara time $\tau \in [0, \beta]$ with β the inverse temperature; see e.g. [14]), and the functional integral defining the NC QFT model [11] can be obtained as a limit $\beta \rightarrow 0$ from that. Thus the model \mathcal{H} in (1) defines a $2n + 1$ dimensional QFT. It would be interesting to use this relation to defer from my results on the latter model results for the former model.

The NC QFT model S in (1) has interesting and non-trivial QFT divergences which one has to treat by regularization and renormalization. I believe that this models provides an interesting example where the role of such divergences can be studied in detail and by exact and explicit results beyond perturbation theory, and that our previous results on this [10, 11] are only a first step in this direction: as I will argue below, there are other QFT limits than the ones studied in this latter work, and one of these other limits is more interesting and more difficult than the others. The explicit solution of the model in this latter limit is a challenging but doable project for the future.

To be more specific: The natural regularization for the NC QFT model S in (5) is to restrict the fields $A_{\ell m}^{(\dagger)}$ to $\ell = (\ell_1, \ell_2, \dots, \ell_n)$ such that

$$\ell_j = 1, 2, \dots, L < \infty \quad (36)$$

and similarly for m . Then the fields $A_{\ell m}$ can be naturally interpreted as components of a ³ $N \times N$ matrix A with $N = L^n$. With that the functional integral defining the NC QFT model becomes a well-defined integral over \mathbb{R}^{2N^2} , and the non-trivial task is to find a dependence of the model parameters σ , $\tilde{\sigma}$, $-\mu$, and g on the cut-off parameter N such that the limit $N \rightarrow \infty$ is well-defined an non-trivial.

In [11] we studied two such limits for the case $\tilde{\sigma} = 0$ which we called IR- and UV limit: The IR limit corresponds to the following scaling of parameters,

$$\sigma = 1, \quad g = \frac{g_{\text{ren}}}{N}, \quad B = \frac{B_{\text{ren}}}{N^{1/n}}, \quad \mu = \mu_{\text{ren}} \quad (37)$$

where the parameters with the subscript “ren” (short for “renormalized”) are independent of N . The results for the Green’s functions in this limit can be found in [11]. I only mentioned here that $B \rightarrow 0$ for $N \rightarrow \infty$ leads to a 2-point Green’s function which is translational invariant, $G(x; y) = G(x - y; 0)$, and all the higher Green’s functions are trivial. In the IR limit the duality symmetry of this model [7] is broken, which implies the existence of a dual limit where $B \rightarrow \infty$ as $N \rightarrow \infty$ and with Green’s function obtained from the ones in the IR limit by a duality transformation. The 2-point Green’s function in this latter UV limit is non-trivial and ultra-local, $G(x; y) \propto \delta^{2n}(x - y)$, and the higher Green’s functions are again trivial [11].

³Note that the symbol N in the following and in sections 1-4 have different meanings!

Now comes my addendum to [11]: It is possible to get a third limit in which the above mentioned duality symmetry is not broken as follows: rather than keeping σ constant and scaling B like $N^{-1/n}$ one can scale σ like $N^{-1/n}$ and keep B constant in the limit $N \rightarrow \infty$:

$$\sigma = \frac{\sigma_{ren}}{N^{1/n}}, \quad g = \frac{g_{ren}}{N}, \quad B = B_{ren}, \quad \mu = \mu_{ren}. \quad (38)$$

The non-trivial scaling of σ can be interpreted as multiplicative regularization. It is easy to deduce from the results in [11] the 2-point Green's function in this third limit,

$$G(x; y) = \sum_{\ell, m} \langle A_{\ell m}^\dagger A_{\ell m} \rangle \phi_{\ell m}(x) \phi_{\ell m}^\dagger(y) \quad (39)$$

with $\langle A_{\ell m}^\dagger A_{\ell m} \rangle$ depending only on ℓ and computed explicitly in [11], section 4.2; the higher Green's functions are again trivial.

I finally would like to emphasize that the limits described above are restricted to the case $\tilde{\sigma} = 0$, and for $\tilde{\sigma} > 0$ there should exist another limit leading to non-trivial higher Green's function and which should describe a non-trivial fixed point of the renormalization group of the models in (1). I expect that this latter limit is the one studied in a closely related model in [17, 18] (this latter model is similar to ours for $\sigma = \tilde{\sigma}$). It is certainly not easy but, as I believe, possible and very desirable to compute the Green's functions of the model in this QFT limit explicitly. In this context I should mention the non-perturbative renormalization of NC ϕ^3 -theory which was recently established by Grosse and Steinacker [19].

Note added: The model \mathcal{H} in (1) for $2n = 2$, $\sigma = \tilde{\sigma} = 1/2$, and at finite temperature was recently studied in [20].

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